Mathisson-Papapetrou-Dixon equations in the Schwarzschild and Kerr backgrounds

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Abstract.

A new representation, which does not contain the third-order derivatives of the coordinates, of the exact Mathisson-Papapetrou-Dixon equations, describing the motion of a spinning test particle, is obtained under the assumption of the Mathisson-Pirani condition in a Kerr background. For this purpose the integrals of energy and angular momentum of the spinning particle as well as a differential relationship following from the Mathisson-Papapetrou-Dixon equations are used. The form of these equations is adapted for their computer integration with the aim to investigate the influence of the spin-curvature interaction on the particle's behavior in the gravitational field without restrictions on its velocity and spin orientation. Some numerical examples for a Schwarzschild background are presented.

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1. Introduction

In general relativity two main approaches have been developed for the description of spinning particle behavior in a gravitational field. Chronologically, the first one was initiated in 1929 when the usual Dirac equation was generalized for curved space-time [1]. The second, the pure classical (non-quantum) approach, was proposed in 1937 [2]. Later it was shown that in a certain sense the equations from [2] follow from the general relativistic Dirac equation as a classical approximation [3].

The focus of this paper is on the equations of motion of the classical spinning particle, which after [2] were obtained in [4] and in many later papers by different methods. These equations can be written as

$$\frac{D}{ds}\left(mu^\lambda + u_\mu\frac{DS^{\lambda\mu}}{ds}\right) = -\frac{1}{2}u^\pi S^{\rho\sigma}R^\lambda{}_{\pi\rho\sigma}, \tag{1}$$

$$\frac{DS^{\mu\nu}}{ds} + u^\mu u_\sigma\frac{DS^{\nu\sigma}}{ds} - u^\nu u_\sigma\frac{DS^{\mu\sigma}}{ds} = 0, \tag{2}$$

where $u^\lambda \equiv dx^\lambda/ds$ is the particle's 4-velocity, $S^{\mu\nu}$ is the tensor of spin, m and D/ds are, respectively, the mass and the covariant derivative with respect to the particle's proper time s; $R^\lambda{}_{\pi\rho\sigma}$ is the Riemann curvature tensor (units $c = G = 1$ are used); here and in the following, latin indices run 1, 2, 3 and greek indices 1, 2, 3, 4; the signature of the metric (–,–,–,+) is chosen.

Equations (1), (2) were generalized in [5] for the higher multipoles of the test particles and now the set (1), (2) is known as the Mathisson-Papapetrou-Dixon (MPD) equations. Multipolar equations of motion for extended test bodies in general relativity were considered in a recent paper [6] and in this context the importance of the seminal work [2] was pointed out.

The first effects of the spin-gravity interaction following from (1)–(2) were considered in [7] for the Schwarzschild field. According to [7] and in many further publications (a list of them is presented, for example, in [8, 9]) the influence of spin on the particle's trajectory is negligible small for practical registrations. However, in this sense much more realistic are the effects connected with spin precession [10].

An interesting point has been elucidated in [11] concerning the possibility of a static position of a spinning particle outside the equatorial plane of the Kerr source of the gravitational field, on its axis of rotation. In spite of the conclusion that such a situation is not allowed by the MPD equations [11], this question stimulated the investigations of possibilities of some non-static (dynamical) effects connected with the particle's motion relative to a Schwarzschild or Kerr mass outside the equatorial plane [12]. Then it was shown that spinning particles moving with relativistic velocity can significantly deviate from geodesics [13, 14].

Some papers are devoted to the investigation of equilibrium conditions of spinning test particles in the Schwarzschild-de Sitter and Kerr-de Sitter space-times [15], as a development of studying the stationary equilibrium positions of charged particles in Reissner-Nordstrom and Kerr-Newman space-times [16].

While investigating the solutions of equations (1), (2), it is necessary to add a supplementary condition in order to choose an appropriate trajectory of the particle's center of mass. Most often conditions [2, 17]

$$S^{\lambda\nu} u_\nu = 0 \tag{3}$$

or [5, 18]

$$S^{\lambda\nu} P_\nu = 0 \tag{4}$$

are used, where

$$P^\nu = mu^\nu + u_\lambda \frac{DS^{\nu\lambda}}{ds} \tag{5}$$

is the 4-momentum. The condition for a spinning test particle

$$\frac{|S_0|}{mr} \equiv \varepsilon \ll 1 \tag{6}$$

must be taken into account as well [11], where $|S_0| = const$ is the absolute value of spin, r is the characteristic length scale of the background space-time (in particular, for the Kerr metric r is the radial coordinate), and S_0 is determined by the relationship

$$S_0^2 = \frac{1}{2} S_{\mu\nu} S^{\mu\nu}. \tag{7}$$

In general, the solutions of equations (1), (2) under conditions (3) and (4) are different. However, in the post-Newtonian approximation these solutions coincide with high accuracy, just as in some other cases [19, 20]. Therefore, instead of exact MPD equations (1) their linear spin approximation

$$m \frac{D}{ds} u^\lambda = -\frac{1}{2} u^\pi S^{\rho\sigma} R^\lambda{}_{\pi\rho\sigma} \tag{8}$$

is often considered. In this approximation condition (4) coincides with (3) (by condition (3) m in equations (1) is a constant quantity).

According to [21] for a massless spinning particle, which moves with the velocity of light, the appropriate condition is (3). The question is which condition is adequate for the motions of a spinning particle with the nonzero mass if its velocity is close to the velocity of light? To answer this question it is necessary to analyze the corresponding solutions of the exact MPD equations (1), (2) both at condition (3) and (4).

The main purpose of this paper is to consider the exact MPD equations under condition (3) in a Kerr metric. Due to the symmetry of this metric equations (1), (2) have the constants of motion: the particle's energy E and the projection of its angular momentum J_z [22–25]. It is known that in the case of the geodesic equations the analogous constants of motion were effectively used for analyzing possible orbits of a spinless particle in a Kerr space-time [26, 27]. Namely, by the constants of energy and angular momentum the standard form of the geodesic equations, which are the differential equations of the second-order by the coordinates, can be reduced to the differential equations of the first order. Naturally, it is interesting to apply a similar procedure to the exact MPD equations. However, in contrast to the geodesic equations,

the exact MPD equations at the condition (3) contain the third derivatives of the coordinates [28, 29]. Therefore, the application of this procedure to the exact MPD equations is significant.

In this paper in order to obtain a full set of the MPD equations at condition (3), without the third derivatives of the coordinates, some differential relationship following from equations (1), (2) is used. We present this relationship in section 2 in general form, for any metric. Its concrete form in the Kerr metric, together with the expressions for E and J_z is used in section 3 and the full set of the differential equations for the dimensionless quantities connected with the particle's coordinates, velocity and spin is described. The explicit form of these equations are written in the Appendix. We analyze the relationship between u^λ and P^λ at condition (4) in section 4. Section 5 is devoted to some numerical examples. We conclude in section 6.

It is important to note that the very condition (3) arose in a natural fashion in the course of its derivation by different methods [30–32]. Therefore, it is of importance to obtain a representation of the exact MPD equations at this condition in the Kerr metric convenient for their further computer integration.

We point out that the integrals of energy and angular momentum of the MPD equations in a Kerr space-time were effectively used for different purposes in [8, 22–25, 33–38] at condition (4).

2. A relationship following from equations (1)–(3)

In addition to the antisymmetric tensor $S^{\mu\nu}$ in many papers the 4-vector of spin s_λ is used as well, where by definition

$$s_\lambda = \frac{1}{2}\sqrt{-g}\varepsilon_{\lambda\mu\nu\sigma}S^{\nu\sigma} \tag{9}$$

and g is the determinant of the metric tensor, $\varepsilon_{\lambda\mu\nu\sigma}$ is the Levi-Civita symbol. It follows from (7), (9) that $s_\lambda s^\lambda = S_0^2$ and at condition (3) we have $s_\lambda u^\lambda = 0$ (other useful relationships with s_λ following from MPD equations at different supplementary condition can be found, for example, in [9]).

The set of equations (2) contains three independent differential equations and in (3) we have three independent algebraic relationships between $S^{\lambda\nu}$ and u_μ. By (3) the components S^{i4} can be expressed through S^{ki}:

$$S^{i4} = \frac{u_k}{u_4}S^{ki}. \tag{10}$$

So, using (10) the components S^{i4} can be eliminated both from equations (2) and (1). That is, in further consideration one can "forgot" about supplementary condition (3) and deal with the three independent components S^{ik}. However, it is appear that more convenient form of equations (1), (2) is not for S^{ik} but for another 3-component value S_i which is connected with S^{ik} by the simple relationship

$$S_i = \frac{1}{2u_4}\sqrt{-g}\varepsilon_{ikl}S^{kl}, \tag{11}$$

where ε_{ikl} is the spatial Levi-Civita symbol. For example, it is not difficult to check that three independent equations of set (2) in terms of S_i can be written as

$$u_4\dot{S}_i + 2(\dot{u}_{[4}u_{i]} - u^\pi u_\rho \Gamma^\rho_{\pi[4}u_{i]})S_k u^k + 2S_n\Gamma^n_{\pi[4}u_{i]}u^\pi = 0, \tag{12}$$

where a dot denotes usual differentiation with respect to the proper time s, and square brackets denote antisymmetrization of indices; $\Gamma^\rho_{\pi 4}$ are the Christoffel symbols.

The simple calculation shows that the 3-component value S_i has the 3-vector properties relative to the coordinate transformations of the partial form $\hat{x}^i = \hat{x}^i(x^1, x^2, x^3), \quad \hat{x}^4 = x^4$ and in this special sense S_i can be called as a 3-vector. (By the way, in the context of equations (1), (2) firstly the 3-vector of spin was used in [7] with the notation $\mathbf{S} = (S^{23}, S^{31}, S^{12})$). By equations (9)–(11) the relationship between S_i and s_λ is

$$S_i = -s_i + \frac{u_i}{u_4}s_4. \tag{13}$$

Let us consider the first three equations of the subset (1) with the indexes $\lambda = 1, 2, 3$. Multiplying these equations by $S_1, \quad S_2, \quad S_3$ correspondingly and taking into account (9)–(11) we get

$$mS_i\frac{Du^i}{ds} = -\frac{1}{2}u^\pi S^{\rho\sigma}S_j R^j{}_{\pi\rho\sigma} \tag{14}$$

(here the covariant derivative Du^i/ds remains 4D, i.e., is determined according to the Christoffel connection of the 4-dimensional space-time). We stress that in contrast to the each equation from set (1), which contain the third derivatives of the coordinates, relationship (14) does not have these derivatives.

Relationship (14) is an analog of relationship (21) from [9] where the spin 4-vector s_λ is used.

3. On set of exact MPD equations with constants of motion E, J_z for the Kerr metric

In the Boyer-Lindquist coordinates $x^1 = r, \quad x^2 = \theta, \quad x^3 = \varphi, \quad x^4 = t$ the non-zero components of the Kerr metric tensor are

$$g_{11} = -\frac{\rho^2}{\Delta}, \quad g_{22} = -\rho^2,$$

$$g_{33} = -\left(r^2 + a^2 + \frac{2Mra^2}{\rho^2}\sin^2\theta\right)\sin^2\theta,$$

$$g_{34} = \frac{2Mra}{\rho^2}\sin^2\theta, \quad g_{44} = 1 - \frac{2Mr}{\rho^2}, \tag{15}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - 2Mr + a^2, \quad 0 \le \theta \le \pi.$$

In this coordinates the constant values of the particle's energy E and the projection of its angular momentum J_z can be written as [22–25]

$$E = P_4 - \frac{1}{2} g_{4\mu,\nu} S^{\mu\nu}, \tag{16}$$

$$J_z = -P_3 + \frac{1}{2} g_{3\mu,\nu} S^{\mu\nu}. \tag{17}$$

It is convenient to use the dimensionless quantities y_i connected with the particle's coordinates by definition

$$y_1 = \frac{r}{M}, \quad y_2 = \theta, \quad y_3 = \varphi, \quad y_4 = \frac{t}{M}, \tag{18}$$

as well as the quantities connected with its 4-velocity

$$y_5 = u^1, \quad y_6 = Mu^2, \quad y_7 = Mu^3, \quad y_8 = u^4 \tag{19}$$

and the spin components [14]

$$y_9 = \frac{S_1}{mM}, \quad y_{10} = \frac{S_2}{mM^2}, \quad y_{11} = \frac{S_3}{mM^2}. \tag{20}$$

(We underline that in the following, in sections 4, 5 and in the Appendix, the notation y_i^k means that the quantity number i from the set of eleven quantities (18)–(20) is to the k-th power.) In addition, we introduce the dimensionless quantities connected with the particle's proper time s and the constants of motion E , J_z:

$$x = \frac{s}{M}, \quad \hat{E} = \frac{E}{m}, \quad \hat{J} = \frac{J_z}{mM}. \tag{21}$$

Quantities (18), (19) satisfy the four simple equations

$$\dot{y}_1 = y_5, \quad \dot{y}_2 = y_6, \quad \dot{y}_3 = y_7, \quad \dot{y}_4 = y_8, \tag{22}$$

here and in the following a dot denotes the usual derivative with respect to x.

Now we point out other seven nontrivial first-order differential equations for the 11 functions y_i. Namely, the first of them follows directly from equation (14). The second is a result of the covariant differentiation of the normalization condition $u_\nu u^\nu = 1$, that is

$$u_\nu \frac{Du^\nu}{ds} = 0. \tag{23}$$

The third and fourth equations follow from (16) and (17) correspondingly if condition (3) is taken into account. Finally, the last three equations for y_i follow directly from (12). This set of the seven equations is presented in the Appendix. Equations (A.3)–(A.9) together with the four equations from (22) are the full set of the exact MPD equations which describe most general motions of a spinning particle in the Kerr gravitational field without any restrictions on its velocity and spin orientation. We stress that the two equations following from (16) and (17) (see (A.5), (A.6)) contain the quantities $\hat{E}$ and $\hat{J}$ as the parameters proportional to the particle's energy and angular momentum according to notation (21).

Now we recall some features of the solutions of the exact MPD equations under condition (3). It is known that in the Minkowski space-time equations (1)–(3) have, in addition to usual solutions describing the straight worldlines, a set of solutions describing the oscillatory (helical) worldlines [28, 29]. The physical interpretation of these superfluous solutions was proposed by C. Möller [30]. He pointed out that in relativity the position of the center of mass of a rotating body depends on the frame of reference, and condition (3) is common for the so-called proper and non-proper centers of mass [39]. The usual solution describe the motion of the proper center of mass and the helical solutions describe the motions of the set of the non-proper centers of mass. Naturally, in general relativity, when the gravitational field is present, the exact MPD equations (1)–(3) have some superfluous solutions as well. Just to avoid these solutions, instead of (3) condition (4) was used in many papers. In contrast to (3), condition (4) picks out the unique worldline of a spinning particle in the gravitational field. However, the question arises: is this worldline close, in the certain sense, to the usual (non-helical) worldline of equations (1), (2) under condition (3)? It is simple to answer this question if the linear spin approximation is valid, because in this case condition (4) practically coincides with (3). Whereas another situation cannot be excluded *a priori* for the high particle's velocity.

Concerning equations (22), (A.3)–(A.9) we stress that by choosing different values of $\hat{E}$ and $\hat{J}$ for the fixed initial values of y_i one can describe the motions of different centers of mass. Among the set of the pairs $\hat{E}$ and $\hat{J}$ there is the single pair corresponding to the proper center of mass. The possible approaches for finding this pair is a separate subject. One of them was proposed in [40] where a method of separation of some non-oscillatory solutions of the exact MPD equations in the Schwarzschild field was considered. In the next section we shall analyze the possibility of using the expressions for E and J_z from (16), (17) at condition (4) for the same purpose. Note that under condition (4) the concrete values of E and J_z are fully determined by the initial values of the particle coordinates and velocity (or momentum), without its acceleration, in contrast to the case with condition (3).

4. Values $\hat{E}$ and $\hat{J}$ according to condition (4)

Let us check the supposition that the single solutions of equations (1), (2) at the supplementary condition (4), corresponding to the fixed initial values of the coordinates, velocity and spin, is close to those solutions of equations (1), (2) at the condition (3) which describe the motion of the proper center of mass with the same initial values. As we pointed out in section 1, this assumption is justified for the velocities which are not close to the velocity of light. Here we shall consider the situation for any velocity.

Let us write the main relationships following from the MPD equations at condition (4) [22–25]. The mass of a spinning particle m is defined as

$$m = \sqrt{P_\lambda P^\lambda} \tag{24}$$

and m is the integral of motion, that is $dm/ds = 0$. The quantity V^λ is the normalized momentum, where by definition

$$V^\lambda = \frac{P^\lambda}{m}. \tag{25}$$

Sometimes V^λ is called the "dynamical 4-velocity", whereas the quantity u^λ from (1)–(3) is the "kinematical 4-velocity" [8]. As the normalized quantities u^λ and V^λ satisfy the relationships

$$u_\lambda u^\lambda = 1, \quad V_\lambda V^\lambda = 1. \tag{26}$$

There is the important relationship between u^λ and V^λ [22–24]:

$$u^\lambda = N\left[V^\lambda + \frac{1}{2m^2\Delta} S^{\lambda\nu} V^\pi R_{\nu\pi\rho\sigma} S^{\rho\sigma}\right], \tag{27}$$

where

$$\Delta = 1 + \frac{1}{4m^2} R_{\lambda\pi\rho\sigma} S^{\lambda\pi} S^{\rho\sigma}, \tag{28}$$

Now our aim is to consider the explicit form of expression (27) for the concrete case of the Schwarzschild metric, for the particle motion in the plane $\theta = \pi/2$ when spin is orthogonal to this plane (we use the standard Schwarzschild coordinates $x^1 = r, \quad x^2 = \theta, \quad x^3 = \varphi, \quad x^4 = t$). Then we have

$$u^2 = 0, \quad u^1 \neq 0, \quad u^3 \neq 0, \quad u^4 \neq 0, \tag{29}$$

$$S^{12} = 0, \quad S^{23} = 0, \quad S^{13} \neq 0. \tag{30}$$

In addition to (30) by condition (4) we write

$$S^{14} = -\frac{P_3}{P_4} S^{13}, \quad S^{24} = 0, \quad S^{34} = \frac{P_1}{P_4} S^{13}. \tag{31}$$

Using (7), (29)–(31) and the corresponding expressions for the Riemann tensor in the Schwarzschild metric, from (27) we obtain

$$u^1 = NV^1\left(1 + \frac{3M}{r^3} V_3 V^3 \frac{S_0^2}{m^2\Delta}\right), \quad u^2 = V^2 = 0,$$

$$u^3 = NV^3\left[1 + \frac{3M}{r^3}(V_3 V^3 - 1)\frac{S_0^2}{m^2\Delta}\right],$$

$$u^4 = NV^4\left(1 + \frac{3M}{r^3} V_3 V^3 \frac{S_0^2}{m^2\Delta}\right), \tag{32}$$

where M is the mass of a Schwarzschild source. According to (28) we write the expression for Δ as

$$\Delta = 1 + \frac{S_0^2 M}{m^2 r^3}(1 - 3V_3 V^3) \tag{33}$$

(the quantity M in (32), (33) is the mass of a Schwarzschild source). Inserting (33) into (32) we get

$$u^1 = \frac{NV^1}{\Delta}\left(1 + \frac{S_0^2 M}{m^2 r^3}\right),$$

$$u^3 = \frac{NV^3}{\Delta}\left(1 - 2\frac{S_0^2 M}{m^2 r^3}\right),$$

$$u^4 = \frac{NV^4}{\Delta}\left(1 + \frac{S_0^2 M}{m^2 r^3}\right). \tag{34}$$

As in (6), we note

$$\varepsilon = \frac{|S_0|}{mr}, \tag{35}$$

where according to the condition for a test particle it is necessary $\varepsilon \ll 1$. However, in our calculations we shall keep all terms with ε.

The explicit expressions for N we obtain directly from the condition $u_\lambda u^\lambda = 1$ in the form

$$N = \Delta\left[\left(1 + \varepsilon^2\frac{M}{r}\right)^2 - 3V_3 V^3 \varepsilon^2 \frac{M}{r}\times \right.$$
$$\left. \times\left(2 - \varepsilon^2\frac{M}{r}\right)\right]^{-1/2}. \tag{36}$$

Inserting (36) into (34) we obtain the expression for the components V^λ through u^λ $(V^2 = u^2 = 0)$:

$$V^1 = u^1 R\left(1 - 2\varepsilon^2\frac{M}{r}\right),$$

$$V^3 = u^3 R\left(1 + \varepsilon^2\frac{M}{r}\right),$$

$$V^4 = u^4 R\left(1 - 2\varepsilon^2\frac{M}{r}\right), \tag{37}$$

where

$$R = \left[\left(1 - 2\varepsilon^2\frac{M}{r}\right)^2 - 3(u^3)^2\varepsilon^2 Mr\left(2 - \varepsilon^2\frac{M}{r}\right)\right]^{-1/2}. \tag{38}$$

The main feature of relationships (37), (38) is that for the high tangential velocity of a spinning particle the values V^1, V^3, V^4 become imaginary. Indeed, if

$$|u^3| > \frac{1}{\varepsilon\sqrt{6Mr}}, \tag{39}$$

in (38) we have the square root of the negative value. (As writing (39) we neglect the small terms of order ε^2; all equations in this section before (39) and after (40) are strict in ε.) Using the notation for the particle's tangential velocity $u_{tang} \equiv ru^3$ by (39) we write

$$|u_{tang}| > \frac{\sqrt{r}}{\varepsilon\sqrt{6M}}. \tag{40}$$

According to estimates similar to those which are presented in [14] if r is not much greater than M, the velocity value of the right-hand side of equation (40) corresponds to the particle's highly relativistic Lorentz γ-factor of order $1/\varepsilon$.

Probably, this fact that according to (25), (37)–(40) the expressions for the components of 4-momentum P^λ become imaginary (if m in (24) is real) is an evidence that condition (4) cannot be used for the particle's velocity which is very close to the velocity of light. However, this point needs some additional consideration. In any case, relationships (37)–(40) are of importance for authors which investigate solutions of the MPD equations at condition (4). We stress that many papers were devoted to study the planar or circular motions of spinning particles in the Schwarzschild or Kerr space-time at different supplementary conditions [7–14, 22–25, 33–35, 40, 41]. Equations (37)–(40) elucidate the new specific features which arise for the highly relativistic motions.

Another aspect of the connection between the spinning particle's momentum and velocity at condition (4) was considered in [22, 23]. It is pointed out in [22] that there exist a critical distance of minimum approach of a particle to the Kerr source where its velocity becomes space-like with the time-like momentum. Similarly according to conclusion from [23] the velocity of the spinning particle become space-like for sufficiently large gravitational fields and/or spins. In contrast to [22, 23] expressions (37)–(40) describe the situation when the velocity is time-like but the components of momentum become imaginary.

The new result of this section as compare to [22, 23] consists in the conclusion that according to (37)–(40) only tangential component of velocity is important in this case, not the radial one, although the orbit is not necessarily circular.

It is interesting to check the possibility of using the values E and J_z as calculated by (37) for computing spinning particle motions by the equations which are described in the previous section if the particle's tangential velocity is much less than the critical value from the right-hand side of (40). At condition (4) the constants E and J_z for the equatorial motions in the Schwarzschild field can be written as

$$E = P_4 + \frac{1}{2}g_{44,1}S^{14} = mV_4 - \frac{1}{2}g_{44,1}\frac{V_3}{V_4}S^{13}, \tag{41}$$

$$J_z = -P_3 - \frac{1}{2}g_{33,1}S^{13} = -mV_3 - \frac{1}{2}g_{33,1}S^{13}. \tag{42}$$

Using the dimensionless quantities y_i as defined in (18), (19), relationship (37) and the simple expression for S^{13} through S_0 from (41), (42) we obtain

$$\hat{E} = R\left[y_8\left(1-\frac{2}{y_1}\right)\left(1-\frac{2\varepsilon^2}{y_1}\right) + \varepsilon y_7\left(1+\frac{\varepsilon^2}{y_1}\right)\right], \tag{43}$$

$$\hat{J} = R\left[y_1^2 y_7\left(1+\frac{\varepsilon^2}{y_1}\right) + \varepsilon y_1 y_8\left(1-\frac{2}{y_1}\right)\left(1-\frac{2\varepsilon^2}{y_1}\right)\right], \tag{44}$$

where according to (21) we note $\hat{E} = E/m$, $\hat{J} = J_z/mM$. Then for the fixed initial values of the quantities y_1, y_7, y_8 in (43), (44) we have the concrete values of $\hat{E}$ and $\hat{J}$ which can be used for numerical integration of the exact MPD equations. Some examples we shall consider in the next section.

5. Numerical examples

Let us consider some solutions of the exact MPD equations (22), (A.3)–(A.9) for the equatorial particle's motion in the Schwarzschild field with the values $\hat{E}$, $\hat{J}$ from (43), (44). We are interested in the highly relativistic motions when $u^2_{tang} \gg 1$ and, at the same time, $|u_{tang}|$ is much less than right-hand side of (40). All figures 1–8 correspond to the situation when the initial values of the particle coordinates and velocity are given as

$$r(0) = 2.5M, \varphi(0) = 0, u^1(0) = -1, Mu^3(0) = 100 \tag{45}$$

(for the equatorial motions $\theta = \pi/2$ identically, i.e., $u^2 = 0$; $u^4(0)$ is determined from the condition $u_\nu u^\nu = 1$). The small value S_0/Mm is equal to 10^{-6} for figures 1–3 and 6×10^{-5} for figures 4–8. According to (35) for the spin 3-vector we have $S_1 = 0$, $S_3 = 0$ and it follows from (12) that $S_2 = rS_0$. For comparison, we present the corresponding solutions of the geodesic equations with the same initial values of the coordinates and velocity.

Figures 1 and 4 show the dependence r vs. s, with the difference that in figure 1 the graph for a spinning particle practically coincides with the corresponding geodesic graph, whereas according to figure 4 the graph for a spinning particle reveals the oscillatory features. The similar property takes place for the dependence φ vs. s according to figure 5 (the simple linear dependence φ vs. s for a spinning particle with $S_0/Mm = 10^{-6}$ is not presented here). Figures 2, 3, 6, 7 illustrate the clear oscillatory regime for the radial and angular particle's velocity. We point out that the amplitude of the oscillation increases with the value of spin, whereas the frequency of this oscillation decreases. At the same time, according to figure 8 the dependence r vs. the coordinate time t does not reveal the oscillatory features. It is not strange because dt/ds oscillates similarly to dr/ds and $d\varphi/ds$ (the corresponding graph for dt/ds is not presented here for brevity) in such a manner that dr/dt is not oscillatory.

It is known that non-proper centers of mass of a spinning particle oscillate around the proper center of mass [28, 29, 39]. Therefore, one can consider the middle lines of the corresponding oscillatory lines in figures 2–7 as such that present the motions of the proper center of mass. So, when $|u_{tang}|$ is much less than right-hand side of (45), the figures above show that equations (22), (A.3)–(A.9) with the relationships (43), (44) can be used to describe the spinning particle motions in the Schwarzschild space-time in some approximation.

Because of the specific oscillatory regime the corresponding graphs in figures 1–8 are calculated on the limited proper time interval. To illustrate some longer trajectories we shall use the expressions for $\hat{E}$ and $\hat{J}$ which follows from relationships (34)–(43) of paper [40]. We recall that in [40] an approach for separation of some non-oscillatory solutions of the exact MPD equations at condition (3) in the equatorial plane of the Schwarzschild space-time was developed (these solutions describe the orbits closer to circular with $r < 3M$). So, according to our notation $y_1 = r/M$, $\hat{E} = E/m$, $\hat{J} = J_z/mM$, and

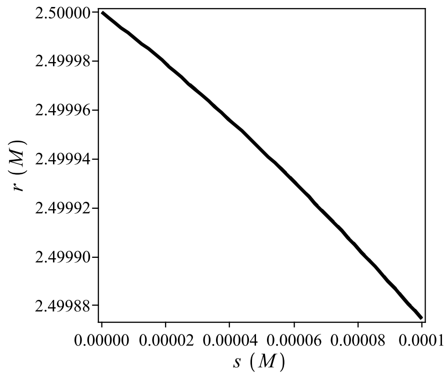

Figure 1. Radial coordinate vs. proper time at $S_0/Mm = 10^{-6}$. The line for the spinning particle practically coincides with the geodesic line at the same initial values (45).

$\varepsilon_0 = S_0/Mm$, now we rewrite the corresponding expressions which are presented in equations (34)–(43) of [40] as

$$Mu^3 = -\frac{1}{\sqrt{\varepsilon_0 y_1}}\left(1-\frac{2}{y_1}\right)^{1/4}\left|1-\frac{3}{y_1}\right|^{-1/2}, \tag{46}$$

$$\hat{E} = \frac{\sqrt{\varepsilon_0}}{\sqrt{y_1}}\left(1-\frac{2}{y_1}\right)^{1/4}\left|1-\frac{3}{y_1}\right|^{-3/2}\left(1-\frac{3}{y_1}+\frac{3}{y_1^2}\right), \tag{47}$$

$$\hat{J} = \sqrt{\varepsilon_0 y_1}\left(1-\frac{2}{y_1}\right)^{-1/4}\left|1-\frac{3}{y_1}\right|^{-3/2}\left(1-\frac{9}{y_1}+\frac{15}{y_1^2}\right). \tag{48}$$

Figures 9–11 correspond to the situation when the initial values of the particle coordinates and velocity are given as

$$r(0) = 2.8M, \varphi(0) = 0, u^1(0) = -0.5 \tag{49}$$

($u^2 = 0$ identically). In addition we put $\varepsilon_0 = 10^{-6}$, then according to (46) we have $Mu^3 \approx -1635$. For comparison in figures 9–11 the corresponding graphs for the geodesic motion are presented. Figures 9–11 illustrate the significant space separation of the trajectories of spinning and spinless particles: for the proper time $s \approx 1.3 \times 10^{-3}M$ (it corresponds to the coordinate time $t \approx 25M$) the spinless particle falls on the horizon surface after less than 0.5 revolution by the angle φ about the Schwarzschild mass, whereas the position of the spinning particle by the radial coordinate is close to the initial value $r = 2.8M$ for one revolution.

We point out that figures 9–11 are similar to same figures from our paper [14]. However, the essential difference must be stressed: all figures in [14] are presented for the linear spin approximation of the MPD equations, whereas here we deal with the exact equations.

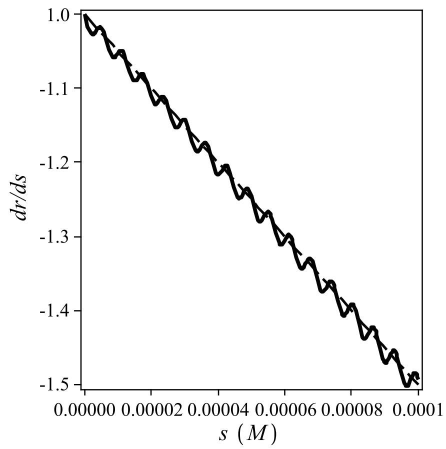

Figure 2. Radial velocity vs. proper time for the spinning particle with $S_0/Mm = 10^{-6}$ (solid line) and for the geodesic motion (dashed line).

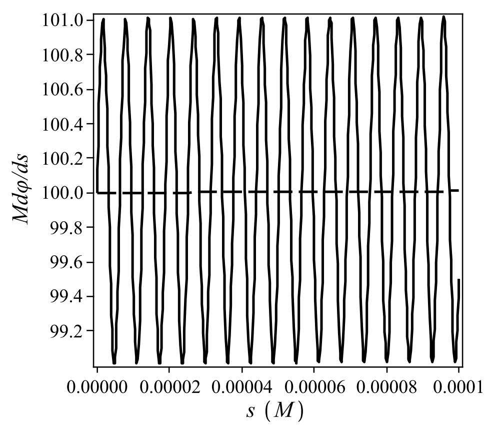

Figure 3. Angular velocity vs. proper time for the spinning particle with $S_0/Mm = 10^{-6}$ (solid line) and for the geodesic motion (dashed line).

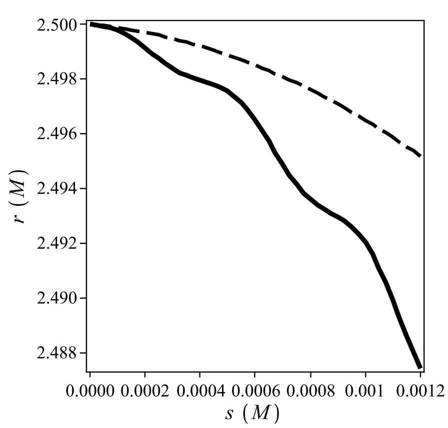

Figure 4. Radial coordinate vs. proper time for the spinning particle with $S_0/Mm = 6 \times 10^{-5}$ (solid line) and for the geodesic motion (dashed line).

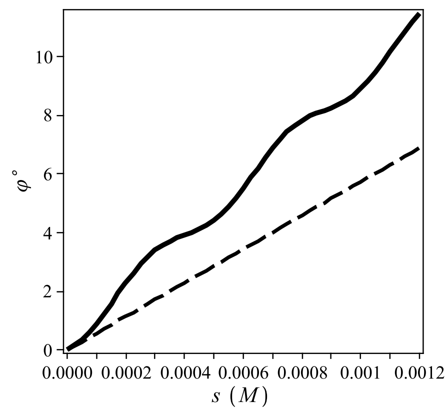

Figure 5. Angle φ vs. proper time for the spinning particle with $S_0/Mm = 6 \times 10^{-5}$ (solid line) and for the geodesic motion (dashed line).

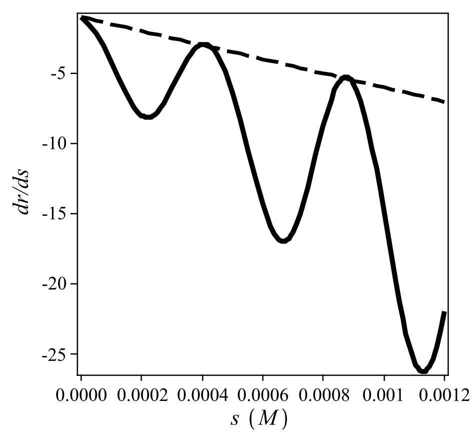

Figure 6. Radial velocity vs. proper time for the spinning particle with $S_0/Mm = 6 \times 10^{-5}$ (solid line) and for the geodesic motion (dashed line).

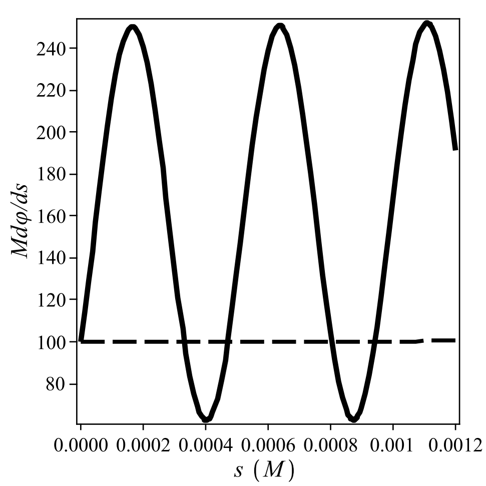

Figure 7. Angular velocity vs. proper time for the spinning particle with $S_0/Mm = 6 \times 10^{-5}$ (solid line) and for the geodesic motion (dashed line).

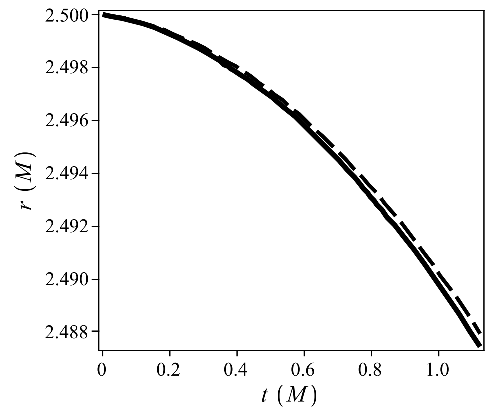

Figure 8. Radial coordinate vs. coordinate time for the spinning particle with $S_0/Mm = 6 \times 10^{-5}$ (solid line) and for the geodesic motion (dashed line).

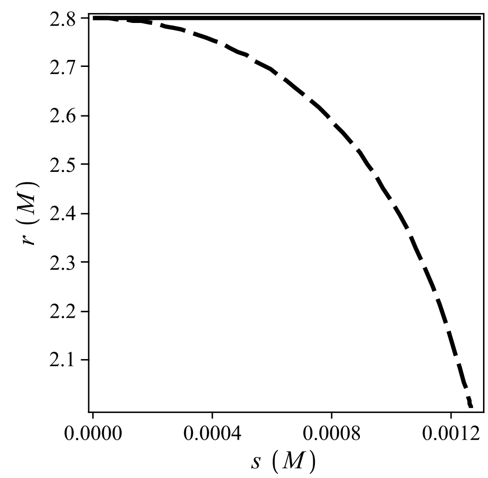

Figure 9. Radial coordinate vs. proper time for the spinning particle with $S_0/Mm = 10^{-6}$ (solid line) and for the geodesic motion with the same initial values of the coordinates and velocity (dashed line).

6. Conclusions

In this paper we obtained the representation of the exact MPD equations at supplementary condition (3) for the Kerr space-time by using the constants of the particle's motion, the energy and angular momentum, together with the differential consequence of these equations (14). The full set of the corresponding 11 first-order differential equations is presented in (22), (A.3)–(A.9). The computer integration of these equations is performed for more simple case of the Schwarzschild space-time. The possibility using expressions (43), (44) to describe motions of a spinning particle in this space-time is considered. Naturally, it is not obvious *a priori* that the constants of motion which are calculated assuming condition (4) can be used for the correct description of the particle motions by the equations which were derived using condition (3). According to figures 1–8 it is possible in some approximation. For a more exact

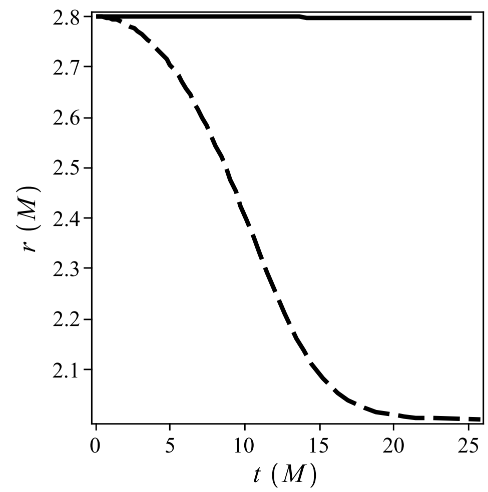

Figure 10. Radial coordinate vs. coordinate time for the spinning particle with $S_0/Mm = 10^{-6}$ (solid line) and for the geodesic motion with the same initial values of the coordinates and velocity (dashed line).

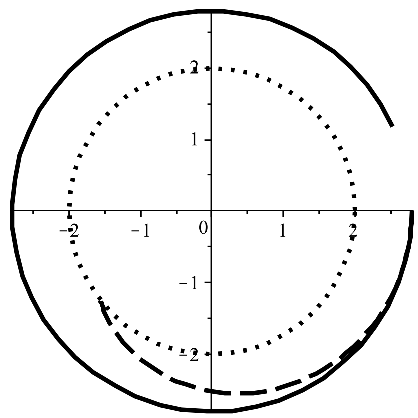

Figure 11. Trajectories in the polar coordinates of the spinning particle with $S_0/Mm = 10^{-6}$ (solid line) and spinless particle (dashed line) with the same initial values of the coordinates and velocity. The circle with the radius 2 corresponds to the horizon line.

description the expressions for $\hat{E}$, $\hat{J}$ from (43), (44) must be corrected. To get these improved values of $\hat{E}$ and $\hat{J}$ one can use of computer search. The suitable values of the pair $\hat{E}$, $\hat{J}$ must give the solutions without large amplitude oscillations. In some cases instead of (43), (44) it is possible to use expressions (47), (48) (figures 9–11).

Independent of the results of sections 2, 3 and 5, the important new information concerning the relationship between the particle's momentum and velocity according to the MPD equations at condition (4) is presented in expressions (37)–(40).

In another paper we plan to carry out a more detailed analysis of the possible choice of values $\hat{E}$, $\hat{J}$ and to present the results of a complex investigation of the highly relativistic motions of a spinning particle in the Kerr space-time according to the exact

MPD equations (22), (A.3)–(A.9).

Appendix. Explicit form of the seven equations described in section 3

First, we stress that the complex explicit form of the seven equations in a Kerr metric, which were discussed in section 3, is determined by the much more complicated expressions for the components of the Riemann tensor and the Christoffel symbols for this metric than for Schwarzschild's one. We do not write these expressions here for brevity because they are presented in other papers, for example, in the Appendix of [8].

We need introduce the notation (in addition to (18)–(21)):

$$z = y_1^2 + \alpha^2 \cos^2 y_2, \quad q = y_1(y_1 - 2) + \alpha^2, \quad \psi = y_1^2 - \alpha^2 \cos^2 y_2,$$
$$\eta = 3y_1^2 - \alpha^2 \cos^2 y_2, \quad \chi = y_1^2 + \alpha^2, \quad \xi = y_1^2 - 3\alpha^2 \cos^2 y_2, \tag{A.1}$$

where α is equal to a/M. One can see that very quantities analogous to (A.1) are presented in the expressions for many components of the Riemann tensor and the Christoffel symbols [8].

To achieve more compact form of the equations we use other notation as well:

$$p_1 = -zy_5q^{-1}, \quad p_2 = -zy_6,$$
$$p_3 = [2\alpha y_1 y_8 \sin^2 y_2 - y_7(z\chi + 2\alpha^2 y_1 \sin^2 y_2)\sin^2 y_2]z^{-1},$$
$$p_4 = [2\alpha y_1 y_7 \sin^2 y_2 + y_8(z - 2y_1)]z^{-1};$$
$$c_1 = y_7 y_{10} \sin y_2 - (z - 2y_1)y_6 y_{11} q^{-1} \sin^{-1} y_2,$$
$$c_2 = y_5 y_{11}(z - 2y_1)q^{-1} \sin^{-1} y_2 - y_7 y_9 q \sin y_2,$$
$$c_3 = (y_6 y_9 q - y_5 y_{10}) \sin y_2;$$
$$d_1 = -(2\alpha q^{-1} y_1 y_6 y_{11} + y_8 y_{10}) \sin y_2,$$
$$d_2 = (2\alpha q^{-1} y_1 y_5 y_{11} + q y_8 y_9) \sin y_2,$$
$$d_3 = (y_5 y_{10} - q y_6 y_9) \sin y_2;$$
$$p = 2\alpha y_1 y_7 \sin^2 y_2 + (z - 2y_1)y_8. \tag{A.2}$$

Then the pointed out in section 3 all seven equations can be written as:

$$y_9\dot{y}_5 + y_{10}\dot{y}_6 + y_{11}\dot{y}_7 = A - y_9 Q_1 - y_{10} Q_2 - y_{11} Q_3, \tag{A.3}$$

$$p_1\dot{y}_5 + p_2\dot{y}_6 + p_3\dot{y}_7 + p_4\dot{y}_8 =$$
$$= -p_1 Q_1 - p_2 Q_2 - p_3 Q_3 - p_4 Q_4, \tag{A.4}$$

$$c_1\dot{y}_5 + c_2\dot{y}_6 + c_3\dot{y}_7 = C - c_1 Q_1 - c_2 Q_2 - c_3 Q_3 + \hat{E}, \tag{A.5}$$

$$d_1\dot{y}_5 + d_2\dot{y}_6 + d_3\dot{y}_8 = D - d_1Q_1 - d_2Q_2 - d_3Q_4 - \hat{J}, \tag{A.6}$$

$$\begin{aligned}
p\dot{y}_9 = {} & \alpha y_5y_7y_9(\alpha^2 - y_1^2)q^{-1}\sin^2 y_2 - \alpha^3 y_1y_6y_7y_9z^{-1}\sin^2 y_2\sin 2y_2 + \alpha^2 y_5y_8y_9(y_1-1)q^{-1} \\
& \times\sin^2 y_2 - 0.5\alpha^2 y_6y_8y_9(z-2y_1)z^{-1}\sin 2y_2 + \alpha y_5^2 y_{11}\psi q^{-2} + 2\alpha y_1y_7^2y_{11}(y_1z(z-2y_1) \\
& -\alpha^2\psi\sin^2 y_2)z^{-2}q^{-1}\sin y_2^2 + \alpha y_8^2y_{11}(z-2y_1)\psi z^{-2}q^{-1} + \alpha y_1y_5y_7y_{10}q^{-1}\sin 2y_2 + 0.5\alpha^2 y_5 \\
& \times y_8y_{10}(z-4y_1)z^{-1}q^{-1}\sin 2y_2 - 2\alpha y_1y_5y_6y_{11}q^{-1}coty_2 + 2\alpha y_1^2y_6y_7y_{10}z^{-1}\sin^2 y_2 + y_1y_6y_8 \\
& \times y_{10}(z-2y_1)z^{-1} + y_7y_8y_{11}(y_1z(z-2y_1)^2 - \alpha^2\psi(z-4y_1)\sin^2 y_2)z^{-2}q^{-1} + \beta[2\alpha y_5^2y_7(q(z \\
& -3y_1^2) + y_1z(y_1-1))q^{-2}\sin^2 y_2 + y_5^2y_8q^{-2}(3q\psi + \alpha^2 z(1-y_1)\sin^2 y_2) + 2\alpha y_1y_5y_6y_7 \\
& \times(z+2\alpha^2\sin^2 y_2)q^{-1}\sin 2y_2 + \alpha^2 y_5y_6y_8(z-4y_1)q^{-1}\sin 2y_2 + 2\alpha y_1^2y_6^2y_7\sin^2 y_2 \\
& +y_1y_6^2y_8(z-2y_1) - y_8^3(z-2y_1)\psi z^{-2} + 2\alpha y_7y_8^2\psi(z-3y_1)z^{-2}\sin^2 y_2 + 2\alpha y_1y_7^3[y_1z\chi(z \\
& -2y_1) + \alpha^2\sin^2 y_2(zq - 2\alpha^2 y_1^2\sin^2 y_2 + 4y_1^3)]z^{-2}q^{-1}\sin^4 y_2 + 2\alpha y_1z(y_7\dot{y}_5 - y_5\dot{y}_7)q^{-1}\sin^2 y_2 \\
& +z(z-2y_1)(y_5\dot{y}_8 - y_8\dot{y}_5)q^{-1} + y_7^2y_8(y_1z^2(z-2y_1) - \alpha^2\psi(z-6y_1)\sin^2 y_2)z^{-2}\sin^2 y_2],
\end{aligned} \tag{A.7}$$

$$\begin{aligned}
p\dot{y}_{10} = {} & \alpha y_5y_6y_{11}(2y_1^2 - z)q^{-1} - 2\alpha^3 y_1y_5y_7y_9z^{-1}\sin^3 y_2\cos y_2 + 0.5\alpha^2 y_5y_8y_9(2y_1-z)z^{-1} \\
& \times\sin 2y_2 + 2\alpha y_1^2y_5y_7y_{10}z^{-1}\sin^2 y_2 + y_1y_5y_8y_{10}(z-2y_1)z^{-1} - 2\alpha y_1y_6^2y_{11}coty_2 \\
& +\alpha y_1y_7^2y_{11}(z+2\alpha^2 y_1\sin^2 y_2)z^{-2}\sin 2y_2 + 2\alpha y_1y_8^2y_{11}(2y_1-z)z^{-2}coty_2 + \alpha y_6y_7y_9q \\
& \times(z-4y_1^2)z^{-1}\sin^2 y_2 + y_6y_8y_9q(4y_1^2 - z(y_1+1))z^{-1} + \alpha y_1y_6y_7y_{10}\sin 2y_2 - 0.5\alpha^2 y_6y_8y_{10} \\
& \times\sin 2y_2 + y_7y_8y_{11}(z^3 + 2y_1z(\alpha^2\sin^2 y_2 - z) - 8\alpha^2 y_1^2\sin^2 y_2)z^{-2}\cot y_2 + \beta[0.5\alpha^2 y_5^2y_8(2y_1 \\
& -z)q^{-1}\sin 2y_2 - 2\alpha^3 y_1y_5^2y_7q^{-1}\sin^3 y_2\cos y_2 - 2\alpha y_5y_6y_7\eta\sin^2 y_2 + 2y_5y_6y_8(4y_1^2 - z(y_1+1)) \\
& +\alpha y_1y_6^2y_7(2z + 3\alpha^2\sin^2 y_2)\sin 2y_2 + \alpha^2 y_6^2y_8(z-6y_1)\sin y_2\cos y_2 + 2\alpha y_1z(y_6\dot{y}_7 - y_7\dot{y}_6) \\
& \times\sin^2 y_2 + z(z-2y_1)(y_6\dot{y}_8 - y_8\dot{y}_6) + \alpha^2 y_1y_8^3(z-2y_1)z^{-2}\sin 2y_2 + 2\alpha y_1y_7^3(\chi z^2 \\
& +4\alpha^2 y_1z\sin^2 y_2 + 2\alpha^4 y_1\sin^4 y_2)z^{-2}\sin^3 y_2\cos y_2 + 2\alpha y_1y_7y_8^2(\alpha^2 y_1\sin^2 y_2 - \chi(z-2y_1))z^{-2} \\
& \times\sin 2y_2 + 0.5y_7^2y_8(z^3q + 2\alpha^2 y_1(zq - 6y_1\chi)\sin^2 y_2)z^{-2}\sin 2y_2],
\end{aligned} \tag{A.8}$$

$$\begin{aligned}
p\dot{y}_{11} = {} & -2\alpha^3 y_1y_6y_7y_{11}z^{-1}\sin^3 y_2\cos y_2 + \alpha y_5y_7y_{11}[(4y_1^2 - z)\chi - 2\alpha^2 y_1^2\sin^2 y_2 - 4y_1^3]z^{-1}q^{-1} \\
& \times\sin^2 y_2 + y_5y_8y_{11}[y_1(z-2y_1)^2 + \alpha^2(z-2y_1^2)\sin^2 y_2]z^{-1}q^{-1} + y_6y_8y_{11}[z(z-2y_1) + 2\alpha^2 y_1 \\
& \times\sin^2 y_2]z^{-1}\cot y_2 - \alpha y_7^2y_9q(\alpha^2\psi + y_1^2(z+2y_1^2))z^{-2}\sin^4 y_2 + y_7y_8y_9q(zy_1(2y_1-z) + \psi(\chi \\
& +\alpha^2\sin^2 y_2))z^{-2}\sin^2 y_2 - \alpha y_8^2y_9q\psi z^{-2}\sin^2 y_2 + 2\alpha^3 y_1y_7^2y_{10}qz^{-2}\sin^5 y_2\cos y_2 - 0.5y_7y_8y_{10}q
\end{aligned}$$

$$\times(z^2+4\alpha^2 y_1\sin^2 y_2)z^{-2}\sin 2y_2+\alpha y_1 y_8^2 y_{10} q z^{-2}\sin 2y_2+\beta[-2\alpha y_5 y_7^2(\chi\psi+2y_1^2 z)z^{-1}\sin^4 y_2$$
$$+4\alpha^3 y_1 y_6 y_7^2 q z^{-1}\sin^5 y_2\cos y_2-2\alpha y_5 y_8^2\psi z^{-1}\sin^2 y_2+2\alpha y_1 y_6 y_8^2 q z^{-1}\sin 2y_2+zq(y_7\dot{y}_8$$
$$-y_8\dot{y}_7)\sin^2 y_2-y_6 y_7 y_8 z q\sin 2y_2-2y_5 y_7 y_8(y_1 z(z-2y_1)-\psi(\chi+\alpha^2\sin^2 y_2))z^{-1}\sin^2 y_2], \qquad (A.9)$$

where

$$A=-2\alpha y_5 y_6 y_9 y_{10}\eta z^{-3}\cos y_2-2\alpha y_5 y_7 y_9 y_{11}\eta z^{-4}(\chi+2\alpha^2\sin^2 y_2)\cos y_2$$
$$-6\alpha y_1 y_6 y_7 y_9 y_{11} q\xi z^{-4}\sin y_2+6y_1 y_6 y_8 y_9 y_{11} q\xi z^{-4}\sin^{-1} y_2+6\alpha y_1 y_7^2 y_9 y_{10} q\xi\chi z^{-5}\sin^3 y_2$$
$$-6y_1 y_7 y_8 y_9 y_{10} q\xi z^{-5}(\chi+\alpha^2\sin^2 y_2)\sin y_2+6\alpha y_1 y_8^2 y_9 y_{10} q\xi z^{-5}\sin y_2$$
$$+2\alpha y_6 y_7 y_{10} y_{11}\eta z^{-4}(2\chi+\alpha^2\sin^2 y_2)\cos y_2-6\alpha y_1 y_5 y_7 y_{10} y_{11}\xi\chi q^{-1} z^{-4}\sin y_2$$
$$+6\alpha^2 y_1 y_5 y_8 y_{10} y_{11}\xi q^{-1} z^{-4}\sin y_2+\alpha y_6^2 y_9^2 q\eta z^{-3}\cos y_2$$
$$+\alpha y_7^2 y_9^2\eta q(\chi^2+2q\alpha^2\sin^2 y_2)z^{-5}\sin^2 y_2\cos y_2$$
$$-2\alpha^2 y_7 y_8 y_9^2\eta q(3\chi-4y_1)z^{-5}\sin^2 y_2\cos y_2+\alpha\eta q(2q+\alpha^2\sin^2 y_2)z^{-5} y_8^2 y_9^2\cos y_2$$
$$+\alpha\eta y_5^2 y_{10}^2 q^{-1} z^{-3}\cos y_2-\alpha\eta(2\chi^2+\alpha^2 q\sin^2 y_2)y_7^2 y_{10}^2 z^{-5}\sin^2 y_2\cos y_2$$
$$-\alpha\eta(q+2\alpha^2\sin^2 y_2)y_8^2 y_{10}^2 z^{-5}\cos y_2+2\alpha^2\eta(3\chi-2y_1)\times y_7 y_8 y_{10}^2 z^{-5}\sin^2 y_2\cos y_2$$
$$+\alpha\eta(q+2\alpha^2\sin^2 y_2)y_5^2 y_{11}^2 q^{-2} z^{-3}\sin^{-2} y_2\cos y_2-\alpha\eta(2q+\alpha^2\sin^2 y_2)$$
$$\times y_6^2 y_{11}^2 q^{-1} z^{-3}\sin^{-2} y_2\cos y_2+6\alpha\xi y_1 y_5 y_6 y_{11}^2 q^{-1} z^{-3}\sin^{-1} y_2$$
$$-4\alpha^3 y_1^2\eta y_7^2 y_{11}^2 q^{-1} z^{-5}\sin^2 y_2\cos y_2$$
$$-2\alpha^2 y_1\eta(q-\alpha^2\sin^2 y_2)(1+\sin^2 y_2)y_7 y_8 y_{11}^2 q^{-1} z^{-5}\cos y_2$$
$$-\alpha\eta(z-2y_1)(q-\alpha^2\sin^2 y_2)y_8^2 y_{11}^2 q^{-1} z^{-5}\cos y_2; \qquad (A.10)$$

$$Q_1=(y_1 q-z(y_1-1))y_5^2 z^{-1} q^{-1}-q y_1 y_6^2 z^{-1}-q(y_1 z^2-\alpha^2\psi\sin^2 y_2)y_7^2 z^{-3}\sin^2 y_2$$
$$+q\psi y_8^2 z^{-3}-\alpha^2 y_5 y_6 z^{-1}\sin 2y_2-2\alpha q\psi y_7 y_8\, z^{-3}\sin^2 y_2,$$
$$Q_2=-0.5\alpha^2 y_6^2 z^{-1}\sin 2y_2+0.5\alpha^2 y_5^2 z^{-1} q^{-1}\sin 2y_2-0.5y_7^2(z^2\chi$$
$$+2\alpha^2 y_1(\chi+z)\sin^2 y_2)z^{-3}\sin 2y_2-\alpha^2 y_1 y_8^2 z^{-3}\sin 2y_2+2y_1 y_5 y_6 z^{-1}+2\alpha y_1 y_7 y_8\chi z^{-3}\sin 2y_2,$$
$$Q_3=2y_5 y_7(y_1 z(z-2y_1)-\alpha^2\psi\sin^2 y_2)z^{-2} q^{-1}+2\alpha y_5 y_8\psi z^{-2} q^{-1}$$
$$+2y_6 y_7(z^2+2\alpha^2 y_1\sin^2 y_2)z^{-2}\cot y_2-4\alpha y_1 y_6 y_8 z^{-2}\cot y_2.$$
$$Q_4=-2\alpha y_5 y_7(2y_1^2 z+\psi\chi)z^{-2} q^{-1}\sin^2 y_2+2y_5 y_8\psi\chi z^{-2} q^{-1}$$
$$+2\alpha^3 y_1 y_6 y_7 z^{-2}\sin^2 y_2\sin 2y_2-2\alpha^2 y_1 y_6 y_8 z^{-2}\sin 2y_2; \qquad (A.11)$$

$$C=-(1-2y_1 z^{-1})y_8-2\alpha y_1 y_7 z^{-1}\sin^2 y_2$$
$$+2\alpha^2 q y_1 y_7 y_9 z^{-3}\sin^2 y_2\cos y_2-2\alpha q y_1 y_8 y_9 z^{-3}\cos y_2$$
$$+\chi\psi y_7 y_{10} z^{-3}\sin y_2-\alpha y_8 y_{10}\psi z^{-3}\sin y_2$$

$$-2\alpha^2 y_1 y_5 y_{11} q^{-1} z^{-2}\cos y_2 - y_6 y_{11}\psi z^{-2}\sin^{-1} y_2; \qquad (A.12)$$

$$D = -2\alpha^3 y_1 y_7 y_9 z^{-3}\sin^4 y_2 \cos y_2 + q(z^2 + 2\alpha^2 y_1 \sin^2 y_2)$$
$$\times y_8 y_9 z^{-3}\cos y_2 - \alpha y_7 y_{10}(\chi\psi + 2y_1^2 z) z^{-3}\sin^3 y_2$$
$$-y_8 y_{10}(y_1 z(z-2y_1) - \alpha^2\psi\sin^2 y_2) z^{-3}\sin y_2$$
$$+2\alpha y_1 y_5 y_{11}\chi q^{-1} z^{-2}\cos y_2 + \alpha\psi y_6 y_{11} z^{-2}\sin y_2$$
$$+p^{-1}[2\alpha y_1 y_7^2(z\chi + 2\alpha^2 y_1\sin^2 y_2) z^{-1}\sin^2 y_2 + qzy_7 y_8$$
$$\times(1 - 8\alpha^2 y_1^2 q^{-1} z^{-2}\sin^2 y_2) - 2\alpha y_1(z-2y_1) z^{-1} y_8^2]\sin^2 y_2; \qquad (A.13)$$

$$\beta = y_5 y_9 + y_6 y_{10} + y_7 y_{11}. \qquad (A.14)$$

We stress that both equations (A.3)–(A.9) and expressions (A.10)–(A.13) become much simpler in the case of the Schwarzschild space-time, when $\alpha \equiv a/M = 0$. Then, for example, instead of long expression (A.10) we have

$$A = 6y_1 y_6 y_8 y_9 y_{11} q\xi z^{-4}\sin^{-1} y_2 - 6y_1 y_7 y_8 y_9 y_{10} q\xi z^{-5}(\chi + \alpha^2\sin^2 y_2)\sin y_2. \qquad (A.15)$$

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